Lecture 23 - 11/28 - Taylor Polynomials and approximations.

Second-order Taylor approximation. Back to the problem of locally approximating functions with polynomials, we have seen that if f is of class C^1 in a neighbourhood of x_0, then the function $g(x) = f(x_0) + f'(x_0)(x - x_0)$ is the best linear approximation of f near x_0 in the sense that $f(x) - g(x) = (x - x_0)\varepsilon(x)$ for some function ε satisfying $\lim_{x\to x_0} \varepsilon(x) = 0$.

When a function has a second-order derivative, we can consider approximating it with second-order polynomials. If such an approximation must hold generally for all functions, it must hold in particular for second-order polynomials. In this case, a direct calculation shows that if $f(x)$ is a second-order polynomial, then for any $x_0 \in \mathbb{R}$ and any x,

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2.$$

This suggests that for a general function f such that $f''(x_0)$ exists at a point x_0, the right-hand side above should be the best quadratic polynomial approximating f near x_0. We denote it

$$T_2[f, x_0](x) := (x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2.$$

the **second-order Taylor polynomial** of f at x_0.

Theorem 87. *If f is of class C^2 in a neighborhood of x_0, then one may write for x near x_0*

$$f(x) = T_2[f, x_0](x) + o((x - x_0)^2).$$

(equivalently, the last term could be written as $(x - x_0)^2\eta(x)$, where $\lim_{x\to x_0} \eta(x) = 0$) Conversely, if a quadratic polynomial $P(x)$ is such that $f(x) = P(x) + o((x - x_0)^2)$ for x near x_0, then $P(x) = T_2[f, x_0]$.

Proof. Set $F(x) := f(x) - T_2[f, x_0](x)$, and we must show that for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|x - x_0| < \delta$ implies $|F(x) - F(x_0)| \le |x - x_0|^2\varepsilon$. Note that by construction $F(x_0) = F'(x_0) = F''(x_0) = 0$.

Let $\varepsilon > 0$. Since F'' is continuous at x_0, there exists δ such that $|x - x_0| < \delta$ implies $|F''(x)| < \varepsilon$. Let $x \in (x_0 - \delta, x_0 + \delta)$. Applying the Mean Value Theorem to F, we have

$$F(x) = F(x) - F(x_0) = F'(x_1)(x - x_0),$$

for some x_1 between x and x_0 (in particular, $|x_1 - x_0| \le |x - x_0|$). Appling the MVT again to F', we have

$$F'(x_1) = F'(x_1) - F'(x_0) = F''(x_2)(x_1 - x_0),$$

for some x_2 between x and x_0, in particular, $x_2 \in (x_0 - \delta, x_0 + \delta)$ so $|F''(x_2)| < \varepsilon$. Piecing it all together, we have

$$|F(x)| = |F''(x_2)||x_1 - x_0||x - x_0| < \varepsilon|x - x_0|^2,$$

hence the proof.

We now prove the converse statement. Suppose $P(x)$ is a quadratic polynomial such that $f(x) = P(x) + o((x - x_0)^2)$ near x_0. By the first part of the theorem, we can write

$$T_2[f, x_0](x) - P(x) = (T_2[f, x_0](x) - f(x)) + (f(x) - P(x)) = o((x - x_0)^2),$$

since we are summing two $o((x-x_0)^2)$. Now since $T_2[f,x_0](x) - P(x)$ is a quadratic polynomial, it can be written as $a(x-x_0)^2 + b(x-x_0) + c$, and the only way that $\lim_{x\to x_0} \frac{1}{(x-x_0)^2}(T_2[f,x_0](x) - P(x)) = 0$, is if all coefficients are zero. Therefore $P = T_2[f,x_0]$. $\square$

Conversely, such formulas can be useful to compute derivatives in a fast way: if one can write (by any means other than computing derivatives),

$$f(x) = a + b(x-x_0) + c(x-x_0)^2 + o((x-x_0)^2),$$

then $f(x_0) = a$, $f'(x_0) = b$ and $f''(x_0) = 2c$.

Example 52. *1. Near $x = 0$, $f(x) = \frac{1}{1-x^2} = 1 + x^2 + o(x^2)$. Therefore, $f(0) = 1$, $f'(0) = 0$, and $f''(0) = 2$.*

2. Near $x = 0$, we would like to compute the first two derivatives of $f(x) = \exp(x)(x+2)^2$ at $x = 0$. Since $\exp(0) = \exp'(0) = \exp''(0) = 1$, we can immediately write $\exp(x) = 1 + x + \frac{1}{2}x^2 + o(x^2)$. Therefore,

$$\begin{aligned} f(x) = \exp(x)(x+2)^2 &= (1 + x + \frac{1}{2}x^2 + o(x^2))(4 + 4x + x^2) \\ &= 4 + 8x + 7x^2 + o(x^2), \end{aligned}$$

and we can therefore read immediately that $f(0) = 4$, $f'(0) = 8$ and $f''(0) = 14$.

Higher-order derivatives and Taylor polynomials. We can define differentiability k times as follows: we already treated $k = 1, 2$ and denote $f^{(1)} = f'$ and $f^{(2)} = f''$. If f is k-times differentiable and $f^{(k)}$ is differentiable at x_0, we say that f is $k+1$ times differentiable at x_0 and call $f^{(k+1)}(x_0) = (f^{(k)})'(x_0) = \lim_{h\to 0} \frac{f^{(k)}(x_0+h) - f^{(k)}(x_0)}{h}$. If $f^{(k)}(x)$ exists for some k and for all x on an interval I, we say that f is k **times continuously differentiable** on I (or **of class** C^k on I).

When f is of class C^k in the neighborhood of x_0, we define the n-th Taylor polynomial of f at x_0, by

$$\begin{aligned} T_n[f,x_0](x) &:= f(x_0) + f'(x_0)(x-x_0) + f''(x_0)/2(x-x_0)^2 + \cdots + f^{(n)}(x_0)/n!(x-x_0)^n \\ &= \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k. \end{aligned}$$

Theorem 88 (Taylor's theorem)**.** *If f is of class C^k in a neighborhood of x_0, then*

$$f(x) = T_n[f,x_0](x) + o((x-x_0)^n).$$

Conversely, if a polynomial $P(x)$ of degree n is such that $f(x) = P(x) + o((x-x_0)^n)$ for x near x_0, then $P(x) = T_n[f,x_0]$.

Proof. The proof is a generalization of Theorem 87. $\square$

Note that if f is a polynomial, then for any $x_0 \in \mathbb{R}$ and $x \in \mathbb{R}$, $f(x) = T_n[f,x_0](x)$.

Example 53. *1. If $f(x) = 3x^2 + 2x - 5$, then*

$$T_2[f,1](x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 = 8(x-1) + 3(x-1)^2.$$

2. If $f(x) = \exp(x)$, then for any $n \in \mathbb{N}$, $f^{(n)}(0) = \exp(0) = 1$. Therefore the Taylor polynomial is given by $T_n[\exp,0](x) = \sum_{k=0}^n \frac{x^k}{k!}$.

3. If $f(x) = \sin x$,

$$T_3[\sin,0](x) = \sin 0 + (\sin' 0(x)) + \frac{\sin'' 0}{2}x^2 + \frac{\sin^{(3)} 0}{6}x^3 = x - \frac{x^3}{6}.$$

Taylor Expansions. While Taylor's theorem gives us an explicit expression for the polynomial which best approximates a given function near x_0, this does not mean that one should always compute it by computing derivatives. Many tricks are useful for such computations, a combination of known expansions and algebraic rules, none of which requiring to compute the derivatives of the full function.

Example 54. *For each function below, find a Taylor expansion to order 2 at $x_0 = 0$.*

1. *$f(x) = (\sin x)(x+1)^4$. Knowing that $\sin 0 = \sin'' 0 = 0$ and $\sin' 0 = 1$, we first write a Taylor expansion of $\sin x = x + o(x^2)$, therefore we are left computing*

$$f(x) = (\sin x)(x+1)^4 = (x + o(x^2))(1 + 4x + 6x^2 + o(x^2)) = x + 4x^2 + o(x^2),$$

after expanding and using the rules $x^p o(x^q) = o(x^{p+q})$. Note that all the terms of the form $o(x^3)$ and up are all dumped into $o(x^2)$.

2. *$f(x) = \frac{1}{1-x}$. Here we know that for $|x| < 1$, the geometric series gives $\frac{1}{1-x} = 1 + x + x^2 + \dots$ In particular,*

$$\frac{1}{1-x} = 1 + x + x^2 + \frac{x^3}{1-x} = 1 + x + x^2 + o(x^2).$$

3. *$f(x) = e^{x^2+1}$. One could write the definition of the exponential $e^{x^2+1} = \sum_{k=0}^\infty \frac{(x^2+1)^k}{k!}$, only to realize that each term in the sum will have a contribution to $T_2[f,0]$. On the other hand, one could write*

$$e^{x^2+1} = e \cdot e^{x^2} = e(1 + x^2 + o(x^2)),$$

and thus $e + ex^2$ is the desired answer.

4. *When $x_0 \neq 0$, note that writing a Taylor expansion of $f(x)$ near $x = x_0$ is the same as writing a Taylor expansion of the function $h \mapsto f(x_0 + h)$ near $h = 0$. For example, to write a Taylor expansion of $\frac{1}{x}$ near $x = 2$ is the same as writing an expansion of $\frac{1}{2+h}$ near $h = 0$. For this we exploit a geometric series again:*

$$\frac{1}{2+h} = \frac{1}{2}\frac{1}{1+h/2} = \frac{1}{2}\left(1 - \frac{h}{2} + \frac{h^2}{4} + o(h^2)\right),$$

where in the last equality, we have used the formula $\frac{1}{1+u} = 1 - u + \frac{u^2}{2} + \dots$ whenever $|u| < 1$. Equivalently,

$$\frac{1}{x} = \frac{1}{2} - \frac{x-2}{4} + \frac{(x-2)^2}{8} + o((x-2)^2) \qquad \text{near } x = 2.$$

Exercises for Lecture 23:

1. Compute the Taylor expansions to order 3 for each of the following functions at the points $x_0 = 0$ and $x_0 = 1$:
   (a) $f(x) = (x^2 + 1)^{25}$.
   (b) $f(x) = \frac{x}{x^2+1}$.
   (c) $f(x) = (1 + x + 2x^2)(\sin x)^2$.
   (d) $f(x) = \sqrt{x+1}$.
2. Let f of class C^n near x_0 with $n \ge 1$. Show that $(T_n[f, x_0])'(x) = T_{n-1}[f', x_0](x)$.
3. Reasoning on Taylor polynomials, prove Leibniz' rule: given f, g of class C^n at x_0, then $f \cdot g$ is of class C^n at x_0 and

$$(f \cdot g)^{(n)}(x_0) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x_0) g^{(n-k)}(x_0).$$

   [Hint: $(f \cdot g)^{(n)}(x_0)$ is $n!$ times the coefficient of $(x - x_0)^n$ in the product $T_n[f, x_0](x) T_n[g, x_0](x)$]
4. (a) Show that if $f(x) = 1/(1 + x)$, then $T_n[f, 0](x) = \sum_{k=0}^{n} (-1)^k x^k$.
   (b) Deduce $T_n[g, 0](x)$ for $g(x) = \ln(1 + x)$. [hint: $g'(x) = ?$]
   (c) Find $T_n[g, 0](x)$ if $g(x) = \tan^{-1}(x)$.
5. Let $f(x) = \exp(-\frac{1}{x})$ for $x > 0$ and $f(x) = 0$ for $x < 0$.
   (a) Show that f can be extended into a continuous function at $x = 0$ with the value $f(0) = 0$.
   (b) For $x > 0$, prove by induction that for any $n \in \mathbb{N}$, there exists a polynomial P_n such that $f^{(n)}(x) = P_n(\frac{1}{x}) \exp(\frac{-1}{x})$. Deduce that $\lim_{x \to 0^+} f^{(n)}(x) = 0$ and therefore, $f^{(n)}$ can be extended continuously at $x = 0$ with the value 0.
   (c) Deduce that $T_n[f, 0](x) = 0$ for any $n \in \mathbb{N}$.