Lecture 24 - 11/30 - L'Hôpital's rule and the division problem.

A toy problem: find expansions for ratios of functions. Suppose that for x near $x_0 = 0$[18], a function can be written as

$$h(x) = \frac{a_0 + a_1 x + a_2 x^2 + \dots}{b_0 + b_1 x + b_2 x^2 + \dots}, \qquad \text{with} \quad b_0 \neq 0,$$

and where the "..." indicate that the expansion goes up to some finite precision to be determined. A natural question is: how to obtain the expansion $h(x) = c_0 + c_1 x + c_2 x^2 \dots$? Namely, how to obtain the coefficients c_k from the coefficients a_i and b_j ?

We will answer this for the first two terms, noting that the formula become more and more tedious when looking for further terms.

- To compute $c_0 = \lim_{x\to 0} h(x)$, the numerator and denominator only need an expansion up to order 1, namely,

$$h(x) = \frac{a_0 + o(1)}{b_0 + o(1)} \to \frac{a_0}{b_0} \quad \text{as } x \to 0, \qquad \text{so} \qquad \boxed{c_0 = \frac{a_0}{b_0}}.$$

- To compute c_1, we study $h(x) - c_0$, and since we need $o(x)$ precision on h to extract the $c_1 x$ term, then we need $o(x)$ precision on the numerator and denominator, namely:

$$h(x) = \frac{a_0 + a_1 x + x\varepsilon(x)}{b_0 + b_1 x + x\eta(x)},$$

  then

$$\begin{aligned} h(x) - c_0 = \frac{a_0 + a_1 x + x\varepsilon(x)}{b_0 + b_1 x + x\eta(x)} - \frac{a_0}{b_0} &= \frac{(b_0 a_1 - a_0 b_1)x + x(b_0\varepsilon(x) - a_0\eta(x))}{b_0(b_0 + b_1 x + x\eta(x))} \\ &= \frac{(b_0 a_1 - a_0 b_1)}{b_0^2} x + x\gamma(x), \end{aligned}$$

  where $\lim_{x\to 0}\gamma(x) = 0$. Therefore, $\boxed{c_1 = \frac{b_0 a_1 - a_0 b_1}{b_0^2}}$.

L'Hôpital's rule. The previous calculations, together with Taylor expansions, will help us address the following questions:

1. How to evaluate $\lim_{x\to x_0} \left(\frac{f(x)}{g(x)}\right)$ when both f and g vanish at x_0 ?
2. What about higher-order derivatives ?

We say that a function f **vanishes at order** k at x_0 if

$$f(x_0) = 0, \qquad f'(x_0) = 0, \qquad \dots \qquad f^{(k-1)}(x_0) = 0, \quad \text{and} \quad f^{(n)}(x_0) \neq 0.$$

When f is of class C^k near x_0 and vanishes at order k at x_0, its Taylor expansion at order k reads

$$f(x) = \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + (x - x_0)^k \eta(x), \qquad \lim_{x\to x_0} \eta(x) = 0.$$

[18] If $x_0 \neq 0$, one may just replace x by $x - x_0$ in everything that follows.

Theorem 89 (L'Hôpital's rule)**.** *Suppose that f and g are of class C^n near x_0, that f vanishes of order k at x_0 and g vanishes at order m at x_0, with both $k \le n$ and $m \le n$. Then*

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \begin{cases} \frac{f^{(k)}(x_0)}{g^{(k)}(x_0)} & \text{if } k \ge m \\ \text{does not exist} & \text{if } k < m. \end{cases}$$

In this sense, as l'Hôpital's rule suggests, if numerator and denominator both vanish, we can differentiate both and study the ratio of derivatives, and this process terminates on many occasions.

Proof. From the hypothesis of the theorem, we can write

$$f(x) = \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + (x-x_0)^k \eta(x), \qquad \lim_{x\to x_0} \eta(x) = 0,$$
$$g(x) = \frac{g^{(m)}(x_0)}{m!}(x-x_0)^m + (x-x_0)^m \eta(x), \qquad \lim_{x\to x_0} \eta(x) = 0,$$

where $f^{(k)}(x_0) \neq 0$ and $g^{(m)}(x_0) \neq 0$. Then we immediately have that

$$\frac{f(x)}{g(x)} = \frac{\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + (x-x_0)^k\eta(x)}{\frac{g^{(m)}(x_0)}{m!}(x-x_0)^m + (x-x_0)^m\varepsilon(x)} = \frac{1}{(x-x_0)^{m-k}} \frac{\frac{f^{(k)}(x_0)}{k!} + \eta(x)}{\frac{g^{(m)}(x_0)}{m!} + \varepsilon(x)}.$$

Observing the form of this ratio, we immediately conclude that the limit as $x \to x_0$ does not exists if $k < m$, and the limit exists and equals $\frac{f^{(k)}(x_0)}{g^{(k)}(x_0)}$ if $k = m$. ☐

Example 55.

1. *Find $\lim_{x\to 1} \frac{\ln x}{x-1}$. This can be written as $f(x)/g(x)$ where $f(1) = g(1) = 0$, $f'(1) = 1$ and $g'(1) = 1$, so the limit is 1 by l'Hôpital's rule.*

2. *Find $\lim_{x\to 0} \frac{\cos x - 1}{x^2}$. Differentiating twice numerator and denominator, one arrives at the limit $-1/2$.*

3. *Find $\lim_{x\to 0} \frac{\cos x - 1 + x^2/2}{(\sin x)^4}$. It may be unwise to differentiate until one gets a finite limit. At the level of Taylor polynomials, look for the first nonzero term in the expansions of the numerator and denominator. Namely, from the expansion of cosine, we see that $\cos x - 1 + \frac{x^2}{2} = \frac{x^4}{24} + o(x^4)$, and since $\sin x = x + o(x)$, then one may compute that $(\sin x)^4 = x^4 + o(x^4)$. We can then write*

$$\frac{\cos x - 1 + x^2/2}{(\sin x)^4} = \frac{\frac{x^4}{24} + o(x^4)}{x^4 + o(x^4)} = \frac{1/24 + o(1)}{1 + o(1)} \to \frac{1}{24} \qquad \text{as} \quad x \to 0.$$

Finding higher-order derivatives of indeterminate ratios. Suppose now that we would like to find $(\frac{f}{g})^{(k)}(x_0)$ when f and g vanish at x_0. From Theorem 89, it is clear that if g vanishes at a higher order than f, we will not be able to compute anything. Therefore, let us suppose that f vanishes at at least the same order as g. That is to say, there exists some n such that, near x_0,

$$g(x) = \frac{g^{(n)}(x_0)}{n!}(x-x_0)^n + \dots, \qquad f(x) = \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \dots,$$

with $g^{(n)}(x_0) \neq 0$. In that case, a factor $(x-x_0)^n$ can be factored out and simplified, and the ratio $\frac{f(x)}{g(x)}$ takes the form

$$\frac{f(x)}{g(x)} = \frac{a_0 + a_1(x-x_0) + a_2(x-x_0)^2 \dots}{b_0 + b_1(x-x_0) + b_2(x-x_0)^2 \dots},$$

where for any $k = 0, 1, \dots$, $a_k = \frac{f^{(n+k)}(x_0)}{(n+k)!}$ and $b_k = \frac{g^{(n+k)}(x_0)}{(n+k)!}$. Then finding the derivatives of $\frac{f}{g}$ is equivalent to obtaining the coefficients $c_0, c_1, \dots$ in the expansion

$$\frac{f(x)}{g(x)} = c_0 + c_1(x-x_0) + c_2(x-x_0)^2 \dots,$$

which is nothing but the division problem we saw earlier ! Once the coefficients c_p are found, the derivatives will be given by $\left(\frac{f}{g}\right)^{(p)}(x_0) = p!\ c_p$.

The first term c_0 is exactly given by l'Hôpital's rule, while the second term is given by, after simplifications,

$$c_1 = \frac{a_1 b_0 - a_0 b_1}{b_0^2} = (n+1)\frac{f^{(n+1)}(x_0)g^{(n)}(x_0) - f^{(n)}(x_0)g^{(n+1)}(x_0)}{(g^{(n)}(x_0))^2}.$$

We can then give this as a second theorem.

Theorem 90. *Suppose that f and g are of class C^{n+1} near x_0, that g vanishes at order n at x_0 and f vanishes at least at order n at x_0. Then*

$$\left(\frac{f}{g}\right)'(x_0) = (n+1)\frac{f^{(n+1)}(x_0)g^{(n)}(x_0) - f^{(n)}(x_0)g^{(n+1)}(x_0)}{(g^{(n)}(x_0))^2}.$$

One could write an (increasingly messy) formula for the general derivative, though the formula itself is not as important as understanding how to obtain the coefficients in practice (which is a polynomial division problem, once appropriately set up). In addition, in certain examples, the process can be cut short if one notices certain cancellations.

Example 56. *To find any derivative of $f(x) = \frac{\sin x}{x}$ at $x = 0$, from the Taylor expansion $\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} + \dots$, we immediately get, for $x \neq 0$,*

$$\frac{\sin x}{x} = 1 - \frac{x^2}{6} + \frac{x^4}{120} + \dots$$

therefore one may read the limits of any derivative near $x = 0$ directly from this expansion. This will happen every time the denominator is just a power of x (or of $(x-x_0)$ if $x_0 \neq 0$).

Example 57. *Let $f(x) = \frac{(\ln(x+1))^2}{\cos x - 1 + 2x^3}$ for $x \neq 0$. Find $\lim_{x\to 0} f(x)$ and $f'(0)$ if the function is appropriately defined at $x = 0$ by $f(0) := \lim_{x\to 0} f(x)$.*

Answer: To solve this, we will need the first two nonzero significative terms in the Taylor expansions of the numerator and denominator. By direct computation of the Taylor polynomial of $\ln(x+1)$, we have $\ln(x+1) = x - \frac{x^2}{2} + o(x^2)$ near $x = 0$, so

$$(\ln(x+1))^2 = \left(x - \frac{x^2}{2} + o(x^2)\right)^2 = x^2 - x^3 + o(x^3).$$

On the other hand, $\cos x - 1 + 2x^3 = \frac{-x^2}{2} + 2x^3 + o(x^3)$. *Therefore,*

$$f(x) = \frac{x^2 - x^3 + o(x^3)}{\frac{-x^2}{2} + 2x^3 + o(x^3)} = \frac{1 - x + o(x)}{-1/2 + 2x + o(x)}, \qquad x \neq 0.$$

Out of this form, we can use the polynomial division approach as above to find that

$$\lim_{x \to 0} f(x) = -2, \qquad f'(0) = -6.$$

Exercises for Lecture 24:

1. For each of the following functions defined for $x \neq 0$, find $\lim_{x\to 0} f(x)$ and $f'(0)$ if the function is appropriately defined at $x = 0$. You may use the familiar formulas for the derivatives of sine and cosine:

   (a) $f(x) = \frac{x}{\sin x}$.

   (b) $f(x) = \frac{1-\cos x}{x^2}$.

   (c) $f(x) = \frac{x^2 - x}{\sin x}$.

   (d) $f(x) = \frac{x}{1 - \cos x - \sin x}$.

   (e) $f(x) = \frac{(\ln(1+x))^2}{x^2 + x^3}$.

2. (a) For $f(x) = (1+x)^{\frac{1}{2}}$, find a general formula for $f^{(k)}(0)$ for $k = 0, 1, 2 \dots$, and write down the k-th Taylor polynomial centered at $x_0 = 0$.

   (b) Now if $x_0 > 0$, write a Taylor expansion of order 3 for f centered at x_0 instead of 0. Instead of differentiating again, write $x = x - x_0 + x_0$ and try to extract out of $f(x)$ an expression of the form $(1+h)^{\frac{1}{2}}$ with h a small parameter[19], so as to exploit the series of part (a).

   (c) Repeat part (a) for the function $f(x) = \ln(1+x)$, near $x_0 = 0$.

   (d) Do as in part (b) to find a Taylor expansion at order 3 of $\ln x$ centered at any $x_0 > 0$ (no need to differentiate).

[19] h is proportional to $(x - x_0)$