6 Integration

Lecture 25 - 12/3 - The Riemann integral.

The theory of Riemann integration.

Definition 21 (Subdivision)**.** *Given $a, b \in \mathbb{R}$ with $a < b$, a* **subdivision** *(or partition) of $[a, b]$ is a finite collection of numbers $\sigma = (x_0, x_1, \dots, x_n)$ such that $a = x_0 < x_1 < \dots < x_n = b$. We denote $\mathfrak{S}([a, b])$ the set of all subdivisions of $[a, b]$.*

A subdivision $\sigma = (x_0, \dots, x_n)$ partitions the interval $[a, b]$ into n sub-intervals

$$[x_0, x_1],\ [x_1, x_2],\ \dots,\ [x_{n-1}, x_n].$$

Let a function $f : [a, b] \to \mathbb{R}$ bounded on $[a, b]$ and $\sigma = (x_0, x_1, \dots, x_n)$ a partition of $[a, b]$. For each k, let $M_k = \sup\{f(x), x \in [x_k, x_{k+1}]\}$ and $m_k = \inf\{f(x), x \in [x_k, x_{k+1}]\}$. Moreover, let us call M and m the supremum and infimum of f over $[a, b]$. Associated with a partition σ, we then define the **lower Riemann sum**

$$S^-(f, \sigma) = m_0(x_1 - x_0) + m_1(x_2 - x_1) + \dots + m_{n-1}(x_n - x_{n-1}) = \sum_{k=0}^{n-1} m_k(x_{k+1} - x_k),$$

as well as the **upper Riemann sum**

$$S^+(f, \sigma) = M_0(x_1 - x_0) + M_1(x_2 - x_1) + \dots + M_{n-1}(x_n - x_{n-1}) = \sum_{k=0}^{n-1} M_k(x_{k+1} - x_k).$$

If we use our intuition that if f, nonnegative to fix ideas, is integrable, its integral represents "the area A under the graph $y = f(x)$", then the lower sum $S^-(f, \sigma)$ is the cumulated area of n rectangles covering an area included under that graph, whereas the upper sum $S^+(f, \sigma)$ is the cumulated area of a collection of rectangles whose total surface covers A completely. See Fig. 15.

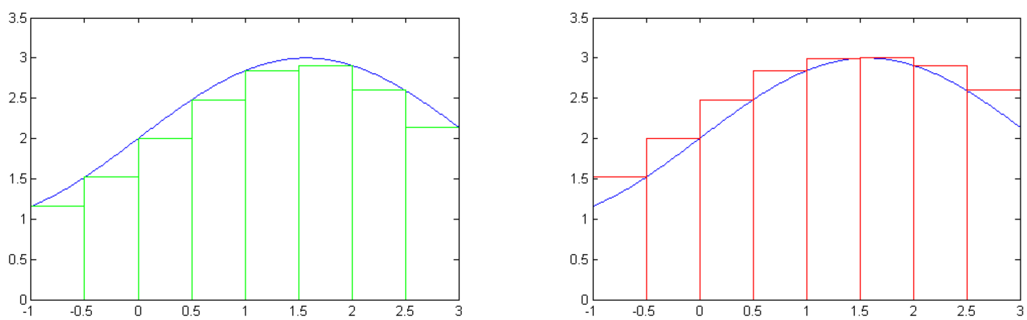

Figure 15: For $\sigma = (-1, -.5, 0., 0.5, 1, 1.5, 2, 2.5, 3)$ and $f(x) = \sin x + 2$, $S^-(f, \sigma)$ is the cumulated area of all green rectangles and $S^+(f, \sigma)$ is the cumulated area of all red rectangles.

Now let us define the sets of upper sums and lower sums

$$U := \{S^+(f, \sigma), \quad \sigma \in \mathfrak{S}\}, \qquad L := \{S^-(f, \sigma), \quad \sigma \in \mathfrak{S}\}.$$

Note that the set of lower sums is bounded above by $M(b-a)$. Indeed, given $\sigma = (x_0, \dots, x_n)$ a subdivision, for any $0 \le k \le n-1$, $m_k \le M$, so we get that

$$S^-(f,\sigma) = \sum_{k=0}^{n-1} m_k(x_{k+1}-x_k) \le M \sum_{k=0}^{n-1}(x_{k+1}-x_k) = M(b-a).$$

Similarly, the set of upper sums U if bounded below by $m(b-a)$. As a result, the following quantities exist

$$I^-(f) := \sup\{S^-(f,\sigma),\ \sigma \in \mathfrak{S}([a,b])\} \qquad \text{and} \qquad I^+(f) := \inf\{S^+(f,\sigma),\ \sigma \in \mathfrak{S}([a,b])\}. \quad (21)$$

The next lemma indicates that any upper sum is always greater than any lower sum, so that in particular, we always have $I^-(f) \le I^+(f)$.

Lemma 91. *For any partitions $\sigma, \sigma' \in \mathfrak{S}([a,b])$, $S^-(f,\sigma) \le S^+(f,\sigma')$.*

Proof. If $\sigma = \sigma'$, the claim follows immediately from the fact that $m_k \le M_k$ on each interval of σ.

If $\sigma \cup \sigma''$, consider the partition $\sigma'' = \sigma \cup \sigma'$, obtained after picking all points in both partitions, throwing out duplicates and arranging the remaining points in increasing order. The claim is that

$$S^-(f,\sigma) \le S^-(f,\sigma'') \le S^+(f,\sigma'') \le S^+(f,\sigma').$$

The middle inequality comes from the previous case, and the other two need justification that, when $\sigma_1 \subset \sigma_2$ then $S^-(f,\sigma_1) \le S^-(f,\sigma_2)$ and $S^+(f,\sigma_1) \ge S^+(f,\sigma_2)$ (i.e., upper sums decrease and lower sums increase when one refines a subdivision). We will prove the left inequality, mainly considering what happens when one adds one point to a subdivision: namely, given two points x and y such that $a \le x < y \le b$ and a third point $z \in (x,y)$, we have

$$\begin{aligned} S^-(f,(x,z,y)) &= (z-x)\min_{[x,z]} f + (y-z)\min_{[z,y]} f \\ &\ge (z-x)\min_{[x,y]} f + (y-z)\min_{[x,y]} f = (y-x)\min_{[x,y]} f = S^-(f,(x,y)). \end{aligned}$$

In general, one may go from σ to σ'' by adding one point at a time, and the lower sums will increase at each step. $\square$

Definition 22 (Riemann-integrability)**.** *A function $f : [a,b] \to \mathbb{R}$ bounded on $[a,b]$ is* **(Riemann-) integrable** *on $[a,b]$ iff $I^-(f) = I^+(f)$. When this is the case, we denote $\int_a^b f(t)\, dt := I^-(f) = I^+(f)$.*

Example 58. *1. A piecewise constant function is integrable. For example, let us show that the function*

$$f(x) = \begin{cases} 1 & \text{if } x \in [0, \frac{1}{2}], \\ 2 & \text{if } x \in (\frac{1}{2}, 1], \end{cases}$$

is integrable with integral $\frac{3}{2}$. Considering the subdvision $\sigma = (0, \frac{1}{2}, 1) = (x_0, x_1, x_2)$, we have that $m_1 = M_1 = 1$, $m_2 = 1$ while $M_2 = 2$. Therefore $S^-(f,\sigma) = 1$ and $S^+(f,\sigma) = \frac{3}{2}$. This implies that $1 \le I^-(f) \le I^+(f) \le \frac{3}{2}$. We now show that in fact, for every $\varepsilon > 0$, there exists a subdivision σ_ε such that $\frac{3}{2} - \varepsilon = S^-(f,\sigma_\varepsilon)$, This will imply that $I^-(f) \ge \frac{3}{2} - \varepsilon$ for every $\varepsilon > 0$, implying that $I^-(f) \ge \frac{3}{2}$ so that, with the previous estimate, $\frac{3}{2} \le I^-(f) \le I^+(f) \le \frac{3}{2}$, *i.e.* $I^-(f) = I^+(f) = \frac{3}{2}$. *Let* $1 >> \varepsilon > 0$ *and consider the subdivision* $\sigma_\varepsilon = (0, \frac{1}{2}, \frac{1}{2} + \varepsilon, 1) = (x_0, x_1, x_2, x_3)$, *then we have* $m_1 = 1, m_2 = 1, m_3 = 2$. *Therefore,*

$$\begin{aligned} S^-(\sigma_\varepsilon, f) &= m_1(x_1 - x_0) + m_2(x_2 - x_1) + m_3(x_3 - x_2) \\ &= 1\left(\frac{1}{2} - 0\right) + 1\left(\frac{1}{2} + \varepsilon - \frac{1}{2}\right) + 2\left(1 - \frac{1}{2} + \varepsilon\right) = \frac{3}{2} - \varepsilon. \end{aligned}$$

Hence the function defined above is integrable with integral $\frac{3}{2}$.

2. $\mathbb{1}_\mathbb{Q}$ *is not integrable on* $[0, 1]$. *Indeed, for any subdivision* σ *of* $[0, 1]$ *and for any subinterval* $[x_{k-1}, x_k]$ *of this subdivision, the sup and inf of* f *are* 1 *and* 0 *respectively (this is because* $\mathbb{Q}$ *and* $\mathbb{R} - \mathbb{Q}$ *are dense in* $[0, 1]$*), so that* $S^-(f, \sigma) = 0$ *and* $S^+(f, \sigma) = 1$ *for all* $\sigma \in \mathfrak{S}([a, b])$. *Therefore,* $I^-(f) = 0 \neq 1 = I^+(f)$, *and* f *is not integrable on* $[0, 1]$.

Monotone functions and continuous functions are integrable on compact intervals. First a characterizing lemma:

Lemma 92 (Characterization of integrability)**.** *If for every* $\varepsilon > 0$, *there exists a subdivision* $\sigma \in \mathfrak{S}$ *such that* $S(f, \sigma) - s(f, \sigma) < \varepsilon$, *then* f *is integrable on* $[a, b]$.

(In fact, the converse is also true)

Proof. We will prove that $I^+(f) \le I^-(f) + \varepsilon$ for every $\varepsilon > 0$, implying as $\varepsilon \to 0$ that $I^+(f) \le I^-(f)$ (since $I^-(f) \le I^+(f)$, this implies $I^-(f) = I^+(f)$). Let $\varepsilon > 0$, by assumption there exists a subdivision σ such that $S^+(f, \sigma) < S^-(f, \sigma) + \varepsilon$ but since $S^+(f, \sigma) \ge I^+(f)$ and $S^-(f, \sigma) \le I^-(f)$ by definition, this implies that $I^+(f) \le^- (f) + \varepsilon$. Hence the proof. □

Theorem 93. *Let* $f : [a, b] \to \mathbb{R}$. *(i) If* f *is monotone on* $[a, b]$, *then* f *is integrable on* $[a, b]$.

(ii) If f *is continuous on* $[a, b]$, *then* f *is integrable on* $[a, b]$.

Proof. In either case, we will use the characterization above, i.e. that if f is either monotone, or continuous, then

$$\forall \varepsilon > 0,\ \exists\, \sigma \in \mathfrak{S}([a, b]), \quad S^+(f, \sigma) - S^-(f, \sigma) < \varepsilon. \tag{22}$$

Proof of (i). Without loss of generality, assume f increasing. Fix $n \in \mathbb{N}$ and choose the equipartition $\sigma = (x_0, x_1, \cdots, x_n)$ with $x_k = a + k\frac{b-a}{n}$. On each interval $[x_k, x_{k+1}]$, since f is increasing, we have that $m_k = f(x_k)$ and $M_k = f(x_{k+1})$. We then compute

$$\begin{aligned} S^+(f, \sigma) - S^-(f, \sigma) = \sum_{k=0}^{n-1} (M_k - m_k)(x_{k+1} - x_k) &= \sum_{k=0}^{n-1} (f(x_{k+1} - f(x_k))\frac{b-a}{n} \\ &= \frac{b-a}{n}(f(b) - f(a)). \end{aligned}$$

Choosing n large enough so that the right-hand side is less than ϵ, we can fulfill the property (22). In particular, f is integrable.

Proof of (ii). Since f is continuous on the compact interval $[a, b]$, it is uniformly continuous there. Let $\varepsilon > 0$, by uniform continuity, there is $\delta > 0$ such that for every $x, y \in [a, b]$ satisfying

$|x-y| < \delta$, $|f(x)-f(y)| < \frac{\varepsilon}{b-a}$. Let $\sigma = (x_0, \ldots, x_n)$ any subdivision such that for every $1 \le k \le n$, $|x_k - x_{k-1}| < \delta$. For every $1 \le k \le n$, f achieves its extrema M_k, m_k at two points x_k' and x_k'', say. Since both points belong to $[x_k, x_{k+1}]$, then $|x_k' - x_k''| < \delta$, so $M_k - m_k = |f(x_k') - f(x_k'')| < \frac{\varepsilon}{b-a}$. Summing this inequality for all k, we obtain

$$\begin{aligned} S^+(f,\sigma) - S^-(f,\sigma) &= \sum_{k=0}^{n-1} (M_k - m_k)(x_{k+1} - x_k) \\ &\le \frac{\varepsilon}{b-a} \sum_{k=0}^{n-1} (x_{k+1} - x_k) = \frac{\varepsilon}{b-a}(b-a) = \varepsilon, \end{aligned}$$

hence the claim (22) is proved. □

Remark 8. *More generally, functions which are continuous outside of finitely (or even, countably) many jump discontinuities are integrable.*

Remark 9. *The method of Riemann integration is reminiscent of what people used to call the computation of integrable "by quadrature" (approximation with quadrangles). Historically, some beautiful tricks were developped to compute integrals of certain functions using smart choices of subdivision. Nowadays, a typical approach is to link integration and differentiation together via the Fundamental Theorem of Calculus (see next lecture), and use what we know about derivatives to compute some integrals.*

Exercises for Lecture 25:

1. Let $f : [0,1] \to \mathbb{R}$ given by $f(x) = 1$ for $x \neq 1/2$ and $f(1/2) = 0$. Prove that f is Riemann-integrable and that $\int_0^1 f(x)\, dx = 1$.

2. In the proof of Lemma 91, show that for $x < z < y$, we have
$$S^+(f,(x,z,y)) \le S^+(f,(x,y)).$$

3. (computation of $\int_0^B x^2\, dx$ by hand) Let $f(x) = x^2$.

   (a) Show by induction that $\sum_{k=1}^n k^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$ for all $n \ge 1$.

   (b) Using an equipartition of $[0,B]$ with $n+1$ points (call it σ_n), compute $S^\pm(f,\sigma_n)$ explicitly as functions of n.

   (c) Send n to infinity and use squeezing to deduce the value of $\int_0^B x^2\, dx$.